Below I introduce the idea of conversions being undertaken by an arbitrageur who buys, sells and converts optimally. This achieves a number of things:

1. Micro-founds conversions and derives the scaled housing prices as a special case with zero conversion costs. In doing so, I hope it can clarify some of the (unspoken) confusion in the literature about prices of different sizes and about how downsizing the housing stock is actually achieved.
2. Allows for relative price changes but in a way that is tractable. We have analytic solutions for relative prices (without positing separate production functions). There is still a one-to-one mapping from housing demand to house prices.
3. These prices respond to relative demand such that aggregate downsizing will see the price of smaller houses increase relative to larger housing (as is intuitive and apparently in the data). The intuition is that arbitrageurs find it increasingly costly to convert many housing types.
4. This should continue to serve one of our original purposes for introducing conversion costs which is ensuring that construction sector activity (and consequently prices) does not fall too much.

Arbitrage and Housing Conversion with Intermediate Inputs (size-weighted formulation)

Suppose the existence of an arbitrageur who is able to buy, convert, and sell different housing types. We suppose an arbitrageur who is able to take advantage of the need to refurbish many houses at once and so, unlike a household, is able to repackage a measure of the housing stock into houses of different quality types. In our economy we have n housing types indexed by $i = 1, \ldots, n$, ordered by quality $\tilde{h}_1 > \tilde{h}_2 > \cdots > \tilde{h}_n > 0$, with the price of a whole type-i house is p_i^h. In the frictionless benchmark with perfectly convertible housing types, no-arbitrage equalizes price per unit of services, i.e.

$$\frac{p_i^h}{\tilde{h}_i} = \frac{p_j^h}{\tilde{h}_j} \quad \text{for all } i, j \qquad \Longleftrightarrow \qquad p_i^h = \frac{\tilde{h}_i}{\tilde{h}_j} p_j^h. \tag{1}$$

We will now consider the possibility of conversion costs. We suppose that to convert a measure of housing x_{ij} from type-i to type-j, the arbitrageur must buy inputs from the construction sector. Specifically, let $\xi_{ij}(x_{ij})$ is the quantity of intermediate input required for the $i \to j$ conversions. The arbitrageur purchases these inputs at the prevailing cost of construction sector output p^h. Let $B_i \geq 0$ be the number of type-i houses *bought*; $S_i \geq 0$ be the number of type-i houses *sold*. Since the arbitrageur is merely repackaging the existing housing stock into different housing types, they face a feasibility constraint given by

$$g_i(S, B, x): \; \tilde{h}_i S_i - \tilde{h}_i B_i + \sum_{j \neq i} \tilde{h}_i x_{ij} - \sum_{j \neq i} \tilde{h}_j x_{ji} \;\leq\; 0, \qquad i = 1, \ldots, n. \tag{2}$$

This says that the quality-weighted number of houses of type-i sold is less than or equal to the number of houses of type-i bought less the number of type-i converted into other types plus the number of other types converted into type-i.

Problem. The arbitrageur is risk neutral and solves

$$\max_{B,S,x \geq 0} \pi \;=\; \sum_{i=1}^{n} p_i^h (S_i - B_i) \;-\; p^h \sum_{i \neq j} \xi_{ij}(x_{ij}) \quad \text{s.t.} \quad g_i(S, B, x) \leq 0 \; (i = 1, \ldots, n). \tag{3}$$

Let $\mu_i \geq 0$ be the multipliers on (2).

FOC for trades. The Lagrangian is $\mathcal{L} = \sum_i p_i^h(S_i - B_i) - p^h \sum_{i\neq j} \xi_{ij}(x_{ij}) - \sum_i \mu_i g_i$. FOCs with respect to S_i and B_i give

$$p_i^h - \mu_i \tilde{h}_i = 0 \text{ (if } S_i > 0), \qquad -p_i^h + \mu_i \tilde{h}_i = 0 \text{ (if } B_i > 0) \Rightarrow \mu_i = \frac{p_i^h}{\tilde{h}_i} \quad \text{(whenever trades clear balances).} \tag{4}$$

Thus μ_i is the shadow value *per unit of housing* of type i.

FOC for conversion flows. For each $i \neq j$,

$$\partial_{x_{ij}}\mathcal{L}: \; -p^h\,\xi'_{ij}(x_{ij}) \;-\; \mu_i\tilde{h}_i \;+\; \mu_j\tilde{h}_i \begin{cases} = 0 & \text{if } x_{ij} > 0, \\ \leq 0 & \text{if } x_{ij} = 0. \end{cases} \tag{5}$$

Substitute (4) to obtain the pairwise no-arbitrage condition :

$$p_i^h \;\geq\; \frac{\tilde{h}_i}{\tilde{h}_j}\,p_j^h \;-\; p^h\,\xi'_{ij}(x_{ij}), \qquad \text{with equality if } x_{ij} > 0. \tag{6}$$

Importantly, in spite of potential arbitrage gains (i.e.: when $\Delta_{ij} := \frac{\tilde{h}_i}{\tilde{h}_j}p_j^h - p_i^h \neq 0$), it is possible that the presence of conversion costs may mean there is a region where it is not worth the arbitrageur converting housing types. In this case, the stock of that housing type would essentially be fixed and we would need to search for the price that lead to demand equaling this fixed supply. For this reason, it is useful to restrict ourselves to cases where ξ is smooth and differentiable for all $x_{ij} > 0$ and that $\xi'_{ij}(0) = 0$. This has the spirit of an Inada condition and is present simply to avoid this corner solution. With this condition in hand, we can be sure that the arbitrageurs will always exploit price differences and ensure that a demanded change in composition of the housing stock will be met.

Three-type specialisation. With $n = 3$ the six directed links satisfy

$$p_1^h \geq \tfrac{\tilde{h}_1}{\tilde{h}_2}p_2^h - p^h\,\xi'_{12}(x_{12}), \qquad p_1^h \geq \tfrac{\tilde{h}_1}{\tilde{h}_3}p_3^h - p^h\,\xi'_{13}(x_{13}), \tag{7}$$

$$p_2^h \geq \tfrac{\tilde{h}_2}{\tilde{h}_1}p_1^h - p^h\,\xi'_{21}(x_{21}), \qquad p_2^h \geq \tfrac{\tilde{h}_2}{\tilde{h}_3}p_3^h - p^h\,\xi'_{23}(x_{23}), \tag{8}$$

$$p_3^h \geq \tfrac{\tilde{h}_3}{\tilde{h}_1}p_1^h - p^h\,\xi'_{31}(x_{31}), \qquad p_3^h \geq \tfrac{\tilde{h}_3}{\tilde{h}_2}p_2^h - p^h\,\xi'_{32}(x_{32}), \tag{9}$$

with equality on active links. Along an active chain $i \to j \to k$, costs add with the appropriate size scaling via (6).

Quadratic conversion costs: Suppose the input requirement itself is quadratic, $\xi_{ij}(x) = \frac{1}{2}kx^2$ with $k > 0$. Then conversion cost is $p^h\,\xi_{ij}(x)$ and the marginal cost entering (6) is $p^h\,\xi'_{ij}(x) = p^h kx$. Because $\xi'_{ij}(0) = 0$, any strictly positive wedge Δ_{ij} induces the flow $x_{ij} = \Delta_{ij}/(p^h k)$ and satisfies

$$p_i^h = \frac{\tilde{h}_i}{\tilde{h}_j}p_j^h - p^h k x_{ij}. \tag{10}$$

Aside: In addition to the standard reasons for introducing increasing marginal costs, we could also introduce an arguement of the flavour of Greenwald and Gueren (possibly without mentioning them). The underlying housing stock is heterogeneous and some houses are very suitable for conversions while others are difficult. With only a small number of required conversions, arbitrageurs are able to buy up the most suitable and convert them at low cost. As the share of the housing stock requiring conversions grows, arbitrageurs are forced to convert less suitable units at higher cost.

Implications for algorithm: Along the steady-state there is no difference. Along the transition, we would solve for (i) the level of total housing quantity demanded (from the household problem) and (ii) the number of conversions necessary. Together, these imply (a) the total amount of construction output needed (and therefore the unit price of this output); and (b) the relative prices of each housing unit as defined by the equations above. Importantly, this can be done in a single step. In the next iteration, we would then update the aggregate construction output price and relative prices.