State Dependent Pricing in General Equilibrium

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1. Households

The household side of the model looks pretty standard. Households consume and supply labor. They can save through one period nominal bonds, which pay nominal interest rate i_t. The household problem can be written as:

$$\begin{aligned} \max_{c_t, n_t, B_{t+1}} \ & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln c_t - \chi n_t^{\vartheta} \right) \\ \text{s.t.} \quad & c_t + \frac{B_{t+1} - B_t}{P_t} \leq w_t n_t + q_t \bar{K} + \Pi_t + R_t \frac{B_t}{P_t} \end{aligned}$$

We can set this problem using a Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln c_t - \chi n_t^{\vartheta} + \lambda_t \left(w_t n_t + q_t \bar{K} + \Pi_t + (1 + R_t) \frac{B_t}{P_t} - c_t + \frac{B_{t+1}}{P_t} \right) \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow \frac{1}{c_t} = \lambda_t \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Leftrightarrow \chi \vartheta \left(n_t \right)^{\vartheta - 1} = \lambda_t w_t \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta \mathbb{E}_t \left(\lambda_{t+1} \left(1 + R_{t+1} \right) \frac{P_t}{P_{t+1}} \right) \tag{3}$$

We can simplify the first order conditions to obtain:

$$\chi \vartheta \left(n_t \right)^{\vartheta - 1} = \frac{1}{c_t} w_t \tag{4}$$

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left(\frac{1}{c_{t+1}} \left(1 + R_{t+1} \right) \frac{P_t}{P_{t+1}} \right) \tag{5}$$

where (4) is the standard static labor supply condition, and (5) is the consumption Euler equation.

2. Firms

2.1. Final Good Firms

There is a single final good firm which bundles intermediate goods into a final good available from consumption. There is a continuum of intermediate good producers indexed by $i \in [0, 1]$. The final good is CES aggregate of these goods:

$$y_t = \left(\int_0^1 {y_{i,t}}^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \tag{6}$$

Their profit maximization problem is:

$$\max_{y_{i,t}} \; P_t \left(\int_0^1 y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 p_{i,t} y_{i,t} di$$

which result in the following first order condition:

$$P_t \left(\int_0^1 y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}-1} = p_{i,t} \tag{7}$$

This implies that the demand function for the intermediate good j is:

$$y_{i,t} = \left(\frac{p_{i,t}}{P_t} \right)^{-\varepsilon} y_t \tag{8}$$

Noticing that the nominal price of the final good is equal to the sum of prices times the quantities of intermediates, i.e. $P_t y_t = \int_0^1 p_{i,t} y_{i,t} dj$ and using the demand function (8) one obtains a relationship between the final and the intermediate goods price:

$$P_t = \left(\int_0^1 {p_{i,t}}^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{9}$$

2.2. Intermediate Good Firms

Intermediate good firms use labor and capital in production, they are affected by aggregate TFP, and the production technology is Cobb-Douglas:

$$y_{i,t} = a_t \left(k_{i,t}\right)^{1-\mu} \left(n_{i,t}\right)^{\mu} \tag{10}$$

Given the downward sloping demand curve, firms have some market power and can set prices. Taking for a moment this price as given, we can determine the amount of capital and labor they use to produce in each period. We write this problem as a cost minimization because it is independent of the price and the firm will always choose capital and labor to minimize their cost. In short, the problem is to minimize nominal costs subject to the restriction that the firm produces at least as much as it is demanded:

$$\begin{aligned} \min_{k_{i,t}, n_{i,t}} \quad & W_t n_{i,t} + Q_t k_{i,t} \\ \text{s.t.} \quad & a_t \left(k_{i,t}\right)^{1-\mu} \left(n_{i,t}\right)^{\mu} \geq \left(\frac{p_{i,t}}{P_t}\right)^{-\varepsilon} y_t \end{aligned}$$

where W_t is the nominal wage and Q_t is the nominal price of capital. We set the problem using the Lagrangian:

$$\mathcal{L} = -\left(W_t n_{i,t} + Q_t k_{i,t}\right) + \varphi_{i,tj} \left(a_t \left(k_{i,t}\right)^{1-\mu} \left(n_{i,t}\right)^{\mu} - \left(\frac{p_{i,t}}{P_t}\right)^{-\varepsilon} y_t\right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial n_{i,t}} = 0 \Leftrightarrow W_t = \varphi_{i,t} a_t \mu \left(\frac{k_{i,t}}{n_{i,t}}\right)^{1-\mu} \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial k_{i,t}} = 0 \Leftrightarrow Q_t = \varphi_{i,t} a_t \left(1-\mu\right) \left(\frac{k_{i,t}}{n_{i,t}}\right)^{-\mu} \tag{12}$$

$\varphi_{i,t}$ has the interpretation as nominal marginal cost, i.e. how much labor cost need to increase if the firm has to produce one more unit of output. Moreover, it is identical for all firms, that is $\varphi_t = \varphi_{i,t}\ \forall i$. This result follows from (11) and (12) which imply that the capital labor ratio is identical across firms since wage and the rental rate of capital are not firm-specific. In particular, equation (11) says to hire labor up until the point where the wage equal the marginal cost times the marginal product of labor. Equivalently, equation (12) says to rent capital until its price equals the marginal cost times the marginal product of capital.

Conditional on the firm charging a price $p_{i,t}$, we can use these two equilibrium conditions in real terms to get the firm's current real profits in period t:

$$\pi_{j,t} = \frac{p_{i,t}}{P_t} y_{i,t} - \varphi_t a_t \mu \left(\frac{k_{i,t}}{n_{i,t}}\right)^{1-\mu} n_{i,t} - \varphi_t a_t \left(1-\mu\right) \left(\frac{k_{i,t}}{n_{i,t}}\right)^{-\mu} k_{i,t} \tag{13}$$

which after substituting in the demand function can be simplified to:

$$\pi_{j,t} = y_t \left(\left(\frac{p_{i,t}}{P_t}\right)^{1-\varepsilon} - \varphi_t \left(\frac{p_{i,t}}{P_t}\right)^{-\varepsilon}\right) \tag{14}$$

where here φ_t is the real marginal cost.

Now let's turn to the pricing decision. Each period, each firm faces a fixed cost of adjusting its nominal price that is different for each firm and is drawn independently over time from a continuous distribution, $\xi_i \sim G$. Firms adjust their price if the gains from doing so outweigh the costs. In particular, there is a marginal firm that is indifferent to changing its price; thus, within each period, only a fraction ω_{0t} of firms will adjust. These adjusting firms will choose the same price $p_t^\star$ because they face the same demand, their cost structure is identical and the fixed adjustment costs are time independent.

Let us assume that there is a discrete distribution of firms of measure 1 that differ in the period when they last adjusted their price. That is the firms distribution is characterized by $\sum_{j=1}^{J} \theta_{j,t} = 1$ where $\theta_{j,t}$ is the share of one of these firms, which last adjusted their prices j periods ago to $p_{t-j}^\star$. We will call each of these groups of firms vintages. The number of vintages J is determined endogenously and the all firms from vintage J will adjust their prices. For the firms in each of the remaining vintages only a fraction $\alpha_{j,t}$ will adjust their price in period t. Thus, the total fraction of firms that adjust their prices is:

$$\omega_{0,t} = \sum_{j=1}^{J} \alpha_{j,t}\theta_{j,t} \quad \text{with} \quad \alpha_{J,t} = 1 \tag{15}$$

while the corresponding fraction of firms that remain with the price set at $t-j$ is:

$$\omega_{j,t} = \left(1-\alpha_{j,t}\right)\theta_{j,t} \tag{16}$$

These "end-of-period" fractions are useful because they serve as weights. The "beginningof-period" fractions are mechanically related to the "end-of-period" fractions:

$$\theta_{j+1,t+1} = \omega_{j,t} \tag{17}$$

The adjusting decision of a firm is based on three considerations: (*i*) Its value if it adjusts gross of adjustment costs, $v_{0,t}$, (*ii*) its value if it doesn't adjust, $v_{j,t}$, and (*iii*) the current realization of its adjustment costs.

The value of a price adjusting firm is

$$v_{0,t} = \max_{p_t^\star} \left\{ \pi_{0,t} + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1 - \alpha_{1,t+1}) v_{1,t+1} \right] + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \alpha_{1,t+1} v_{0,t+1} \right] - \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \Xi_{1,t+1} \right] \right\} \tag{18}$$

where λ_{t+1}/λ_t is the ratio of future to current marginal utility, which is the appropriate discount factor, and $\mathbb{E}_t \left[(\lambda_{t+1}/\lambda_t) \Xi_{1,t+1} \right]$ represents the present value of next period's adjustment costs.

The value of a firm that maintains its price at $p_{t-j}^\star$ is given by

$$v_{j,t} = \pi_{j,t} + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1 - \alpha_{j+1,t+1}) v_{j+1,t+1} \right] + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \alpha_{j+1,t+1} v_{0,t+1} \right] - \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \Xi_{1,t+1} \right] \tag{19}$$

for $j = 1, \ldots, J-1$. There is no max operator in (17) because the only decision made by a non-adjusting firm is their input mix which already incorporated in the profit term.

The assumption that there is a continuum distribution of fixed costs implies that there will be a marginal firm of each vintage $j = 1, \ldots, J-1$ which is indifferent between adjusting and maintaining its price given its fixed cost realization, i.e. $v_{0,t} - v_{j,t} = W_t \xi$. Thus, this indifference condition determines the fraction of adjusting firms that set the price to $p_t^\star$

$$\alpha_{j,t} = G\left(\frac{v_{0,t} - v_{j,t}}{W_t} \right) \tag{20}$$

where $G(\cdot)$ is the c.d.f. of the fixed-cost distribution with the following properties $G(0) = 0 < G(x) < G(B) = 1$ with $B < \infty$. This condition implies that to induce more firms to adjust the value difference should be larger or the wage should be lower. Given the fraction of firms that adjust and the wage rate, total resources associated with adjustment by the jth vintage are given by

$$\Xi_{j,t} = W_t \int_0^{G^{-1}(\alpha_{j.t})} x g(x) dx \tag{21}$$

where $g(\cdot)$ is the p.d.f. of $G(\cdot)$. It is convenient to assume that $G(\cdot)$ has the following functional form:

$$G(x) = \frac{B\left[\arctan(bx - d\pi) + \arctan(d\pi)\right]}{\arctan(b - d\pi) + \arctan(d\pi)} \tag{22}$$

where B, b, and d are parameters that ensure that (22) is satisfied. As a result, we can re-write (21) as follows:

$$\begin{aligned}
\Xi_{j,t} &= \int_0^{\alpha_{j,t}} G(x)dx = \int_0^{\alpha_{j,t}} \frac{B\left[\arctan(bx - d\pi) + \arctan(d\pi)\right]}{\arctan(b - d\pi) + \arctan(d\pi)} dx = \\
&= \int_0^{\alpha_{j,t}} \frac{B \arctan(bx - d\pi)}{\Omega} dx + \alpha_{j,t} \frac{B \arctan(d\pi)}{\Omega} = \\
&= \frac{B}{\Omega}\left[(bx - d\pi)\arctan(bx - d\pi) - \frac{1}{2}\ln\left(1 + (bx - d\pi)^2\right) + C\right]_0^{\alpha_{j,t}} + \alpha_{j,t}\frac{B\arctan(d\pi)}{\Omega} = \\
&= (b\alpha_{j,t} - d\pi)\arctan(b\alpha_{j,t} - d\pi) - \frac{1}{2}\ln\left(1 + (b\alpha_{j,t} - d\pi)^2\right) - d\pi\arctan(-d\pi) + \\
&\quad + \frac{1}{2}\ln\left(1 + d^2\pi^2\right) + \alpha_{j,t}\frac{B\arctan(d\pi)}{\Omega}
\end{aligned} \tag{23}$$

which implies that for $j = J$ the adjustment cost measure is time-invariant. **Need to check this integral ... am I missing something?**

What is the optimal reset price? The first order condition coming from the dynamic program (18) implies that the optimal price satisfies an Euler equation that involves balancing pricing effects on current and expected profits. That is,

$$0 = \frac{\partial \pi_{0,t}}{\partial p_t^\star} + \beta \mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}(1 - \alpha_{1,t+1})\frac{\partial v_{1,t+1}}{\partial p_t^\star}\right] \tag{24}$$

The derivate of the profit with respect to the optimal reset price is

$$\frac{\partial \pi_{0,t}}{\partial p_t^\star} = y_t\left[(1-\varepsilon)\left(\frac{P_t^\star}{P_t}\right)^{-\varepsilon} + \varepsilon\varphi_t\left(\frac{P_t^\star}{P_t}\right)^{-\varepsilon-1}\right] \tag{25}$$

while the derivative with respect to the non-adjusting value next period imply a recursion:

$$\frac{\partial v_{1,t}}{\partial p^\star_{t-1}} = \frac{\partial \pi_{1,t}}{\partial p^\star_{t-1}} + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1-\alpha_{2,t+1}) \frac{\partial v_{2,t+1}}{\partial p^\star_{t-1}} \right] \tag{26}$$

For now, assume that $J = 2$. Then, we can easily solve for the reset price using (24)-(26). First, note that $\alpha_{2,t+1} = 1$. Thus, (26) can be simplified to

$$\frac{\partial v_{1,t}}{\partial p^\star_{t-1}} = \frac{\partial \pi_{1,t}}{\partial p^\star_{t-1}} \tag{27}$$

Then, plugging (27) into (24) and using the value of derivative of the profit with respect to the optimal reset price we get

$$\begin{aligned} 0 = y_t &\left[(1-\varepsilon) \left(\frac{p^\star_t}{P_t} \right)^{-\varepsilon} + \varepsilon \varphi_t \left(\frac{p^\star_t}{P_t} \right)^{-\varepsilon-1} \right] + \\ &+ \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1-\alpha_{1,t+1}) y_{t+1} \left[(1-\varepsilon) \left(\frac{p^\star_t}{P_{t+1}} \right)^{-\varepsilon} + \varepsilon \varphi_t \left(\frac{p^\star_t}{P_{t+1}} \right)^{-\varepsilon-1} \right] \right] \end{aligned} \tag{28}$$

Taking $(p^\star_t)^{-\varepsilon}$ as a common factor and re-arranging we get:

$$\begin{aligned} y_t(\varepsilon-1)P_t^\varepsilon + \beta \mathbb{E}_t &\left[\frac{\lambda_{t+1}}{\lambda_t} (1-\alpha_{1,t+1}) y_{t+1} (\varepsilon-1) P_{t+1}^\varepsilon \right] = \\ = y_t \varepsilon \varphi_t \frac{1}{p^\star_t} \left(\frac{1}{P_t} \right)^{-\varepsilon-1} &+ \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1-\alpha_{1,t+1}) y_{t+1} \varepsilon \varphi_{t+1} \frac{1}{p^\star_t} \left(\frac{1}{P_t} \right)^{-\varepsilon-1} \right] \end{aligned} \tag{29}$$

Solving for $p^\star_t$

$$p^\star_t = \frac{\varepsilon}{\varepsilon-1} \frac{y_t \varphi_t P_t^\varepsilon + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1-\alpha_{1,t+1}) y_{t+1} \varphi_{t+1} P_{t+1}^\varepsilon \right]}{y_t P_t^{\varepsilon-1} + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1-\alpha_{1,t+1}) y_{t+1} P_{t+1}^{\varepsilon-1} \right]} \tag{30}$$

Need to check exponents of P_t in the denominator ...

In the more general case when there are J vintages, we have that the marginal value is

$$\frac{\partial v_{j,t}}{\partial p^\star_{t-j}} = \frac{\partial \pi_{j,t}}{\partial p^\star_{t-j}} + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} (1-\alpha_{j+1,t+1}) \frac{\partial v_{j+1,t+1}}{\partial p^\star_{t-j}} \right] \tag{31}$$

for $j = 1, \ldots, J-2$ and

$$\frac{\partial v_{J-1,t}}{\partial p^{\star}_{t-J}} = \frac{\partial \pi_{J-1,t}}{\partial p^{\star}_{t-J+1}} \tag{32}$$

since $\alpha_{J,t+1} = 1$. Finally, the marginal profit term for a vintage $j = 1, \ldots, J$ is

$$\frac{\partial \pi_{j,t}}{\partial p^{\star}_{t-j}} = y_t \left[(1-\varepsilon) \left(\frac{P^{\star}_{t-j}}{P_t} \right)^{-\varepsilon} + \varepsilon \varphi_t \left(\frac{P^{\star}_{t-j}}{P_t} \right)^{-\varepsilon-1} \right] \tag{33}$$

Given (31)-(33), one can obtain the following general expression for the reset price:

$$p^{\star}_t = \frac{\varepsilon}{\varepsilon-1} \frac{\sum_{j=0}^{J-1} \beta^j \mathbb{E}_t \left[(\lambda_{t+j}/\lambda_t)(\omega_{j,t+j}/\omega_{0,t}) y_{t+j} \varphi_{t+j} P^{\varepsilon}_{t+j} \right]}{\sum_{j=0}^{J-1} \beta^j \mathbb{E}_t \left[(\lambda_{t+j}/\lambda_t)(\omega_{j,t+j}/\omega_{0,t}) y_{t+j} P^{\varepsilon-1}_{t+j} \right]} \tag{34}$$

where $\omega_{j,t+j}/\omega_{0,t} = (1-\alpha_{j,t+j})(1-\alpha_{j-1,t+j-1}) \ldots (1-\alpha_{1,t+1})$ is the probability of non-adjustment through t to $t+j$. Equation (34) illustrates that the optimal prices varies with current and expected future demands, aggregate price levels, real marginal costs, discount factors and adjustment probabilities. The conditional probability terms are present in time-dependent models, but they are not time-varying. In this set up, they effectively modify the discount factor in a time-varying manner: a very low probability of non-adjustment in some future period leads to the firm to set a price that heavily discounts the effects of profits beyond that period.

3. Aggregation and Market-Clearing

We assume the existence of a central bank that sets money supply according to a money growth rule:

$$\ln M^s_{t+1} - \ln M^s_t = (1-\rho_m)\,\pi^* + \rho_m \left(\ln M^s_t - \ln M^s_{t-1} \right) + \varepsilon_{m,t} \tag{35}$$

We want to express this rule in terms of real money balances since they are stationary. For that purpose, define $m_t = \frac{M_{t+1}}{P_t}$. Add and subtract $\ln P_t$ terms as necessary:

$$\ln M_{t+1}^s - \ln P_t + \ln P_t - \ln P_{t-1} - \ln M_t^s + \ln P_{t-1} = (1-\rho_m)\,\pi^* + \dots$$
$$\dots + \rho_m\left(\ln M_t^s - \ln P_{t-1} + \ln P_{t-1} - \ln P_{t-2} - \ln M_{t-1}^s + \ln P_{t-2}\right) + \varepsilon_{m,t} \tag{36}$$

Define $\pi_t = \ln P_t - \ln P_{t-1}$ and $\Delta \ln m_t = \ln m_t - \ln m_{t-1}$. Then, we can re-write expression (36) as:

$$\Delta \ln m_t^s = (1-\rho_m)\pi^* - \pi_t + \rho_m \Delta \ln m_{t-1}^s + \rho_m \pi_{t-1} + \varepsilon_{m,t} \tag{37}$$

On the demand side, we assume money demand has unit elasticity with respect to consumption and a constant semi-elasticity with respect to the nominal interest rate:

$$\ln(M_{t+1}^d / P_t) = \ln c_t - \eta R_t \tag{38}$$

In equilibrium, all households and firms must be maximizing, all output must be consumed, bond holdings must be in zero net supply and the money and labor market must clear. These conditions can be summarized as follows:

$$c_t = y_t = \left(\int_0^1 y_{i,t}^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}} \tag{39}$$

$$b_t = 0 \tag{40}$$

$$n_t = \int_0^1 n_{j,t} \tag{41}$$

$$m_t^s = m_t^d \tag{42}$$

where n_t must be consistent with the FOC for labor supply.

Individual firms choose their prices according to (18) taken the prices charged by other firms as given, but since they all choose the same price, (9) simplifies to

$$P_t = \left[\sum_{j=0}^{J-1} \omega_{j,t} \left(p_{t-j}^{\star}\right)^{1-\varepsilon}\right]^{1/(1-\varepsilon)} \tag{43}$$