Steady State Derivations

Juan Castellanos

March 2025

1. Mortgage Rates

1.1. Fixed Rate Mortgages

1.1.1. Key equilibrium conditions

$$1 = \Omega^m_{s,t} + \Omega^{x,FRM}_{s,t}\left(q^*_t - \Delta_{q,t}\right) \tag{1}$$

where $\Omega^m_{s,t}$ and $\Omega^{x,FRM}_{s,t}$ are the marginal continuation benefits to the saver of an additional unit of face value and an additional dollar of promised initial payments, respectively. These values are defined by

$$\Omega^m_{s,t} = \mathbb{E}_t\left[\Lambda^s_{t,t+1}\pi^{-1}_{t+1}\left(\rho\,(1-\nu) + (1-\rho)(1-\nu)\Omega^m_{s,t+1}\right)\right] \tag{2}$$

$$\Omega^{x,FRM}_{s,t} = \mathbb{E}_t\left[\Lambda^s_{t,t+1}\pi^{-1}_{t+1}\left(1 + (1-\rho)\,(1-\nu)\,\Omega^{x,FRM}_{s,t+1}\right)\right] \tag{3}$$

where as for the borrower we have used the definition of the saver's stochastic discount factor.

1.1.2. Steady state

Assume without loss of generality that $\pi_{ss} = 1$. Then, saver's marginal continuation benefits in steady state are given by

$$\Omega^m_s = \frac{\beta_s\rho\,(1-\nu)}{1-\beta_s(1-\rho)(1-\nu)} \tag{4}$$

$$\Omega^{x,FRM}_s = \frac{\beta_s}{1-\beta_s(1-\rho)(1-\nu)} \tag{5}$$

Substituting these back into equation (**??**) gives us the steady state value of the mortgage coupon q_{ss}

$$\begin{aligned} q^{*,HRM}_{ss} &= \frac{1-\Omega^m_s}{\Omega^x_s} + \mu_q \\ &= \frac{1-\beta_s(1-\rho)(1-\nu)-\beta_s\rho\,(1-\nu)}{\beta_s} \\ &= \frac{1}{\beta_s} - (1-\nu) + \mu_q\;. \end{aligned} \tag{6}$$

1.2. Adjustable Rate Mortgages

1.2.1. Key equilibrium conditions

$$\Omega_{s,t}^{ARM} = 1 \tag{7}$$

where $\Omega_{s,t}^{ARM}$ is again the total continuation benefit of an additional unit of debt in the ARM economy and it is given by

$$\Omega_{s,t}^{ARM} = \mathbb{E}_t \left[\Lambda_{t,t+1}^{s} \pi_{t+1}^{-1} \left(\left(q_t^* - \Delta_{q,t} \right) + \rho\,(1-\nu) + (1-\nu)(1-\rho)\Omega_{s,t+1}^{ARM} \right) \right] \; . \tag{8}$$

1.2.2. Steady state

The saver's marginal continuation benefit in steady state is given by

$$\Omega_{s}^{ARM} = \frac{\beta_s \left(q_{ss}^* - \mu_q \right) + \beta_s \rho\,(1-\nu)}{1 - \beta_s (1-\rho)(1-\nu)} \tag{9}$$

which after plugging it back to equation (**??**) and rearranging terms one gets:

$$q_{ss}^{*,ARM} = \frac{1}{\beta_s} - (1-\nu) + \mu_q \; . \tag{10}$$

1.3. Hybrid Rate Mortgages

1.3.1. Key equilibrium conditons

$$1 = \Omega_{s,t}^{m} + \Omega_{s,t}^{x} \left(q_t^* - \Delta_{q,t} \right) \tag{11}$$

where $\Omega_{s,t}^{m}$ and $\Omega_{s,t}^{x}$ are the marginal continuation *benefits* of an additional unit of debt and of an additional dollar of initial payments, respectively. Similarly to the borrower's marginal costs, these marginal benefits under HRM contracts have some similarities with their counterparts under FRM and ARM contracts. In fact, it is also the case that the marginal benefit of an additional unit of debt is identical under FRM and HRM contracts. That is,

$$\Omega_{s,t}^{m} = \Omega_{s,t}^{m,FRM} = \Omega_{s,t}^{m,HRM} = \mathbb{E}_t \left[\Lambda_{t,t+1}^{s} \pi_{t+1}^{-1} \left(\rho\,(1-\nu) + (1-\rho)(1-\nu)\Omega_{s,t+1}^{m} \right) \right] \; . \tag{12}$$

Moreover, the marginal benefit of an additional dollar of initial payments under HRM is also equal to its FRM counterpart up to period T when the contract switches to adjustable rates and it is zero afterwards as in the ARM case as shown below:

$$\Omega^{x,HRM}_{s,t} = \sum_{\tau=1}^{T} (1-\rho)^{\tau-1} (1-\nu)^{\tau-1} \mathbb{E}_t \left[\left(\prod_{j=0}^{\tau-1} \Lambda^{s}_{t+j+1,t+j} \pi^{-1}_{t+j+1} \right) \right] . \tag{13}$$

1.3.2. Steady state

We know from the previous derivations that the marginal benefit of an additional unit of debt in the steady state is:

$$\Omega^{m}_{s} = \frac{\beta_s \rho (1-\nu)}{1-\beta_s(1-\rho)(1-\nu)} \tag{14}$$

while the marginal benefit of an additional unit of mortgage payments can be derive from equation (**??**) and simplifies to

$$\Omega^{x,HRM}_{s} = \sum_{\tau=1}^{T} (1-\rho)^{\tau-1} (1-\nu)^{\tau-1} \beta_s^{\tau} . \tag{15}$$

Finally, substituting this two expressions into (**??**) results in

(16)

$$\begin{aligned} q^{*,HRM}_{ss} &= \frac{1-\Omega^{m}_{s}}{\Omega^{x}_{s}} + \mu_q \\ &= \left(1 - \frac{\beta_s \rho(1-\nu)}{1-\beta_s(1-\rho)(1-\nu)}\right) \frac{1}{\sum_{\tau=1}^{T} (1-\rho)^{\tau-1} (1-\nu)^{\tau-1} \beta_s^{\tau}} + \mu_q \\ &= \frac{1}{\beta_s \sum_{\tau=0}^{T-1} \left[(1-\rho)(1-\nu)\beta_s\right]^{\tau}} - \frac{\rho(1-\nu)}{1-\beta_s(1-\rho)(1-\nu)} \frac{1}{\sum_{\tau=0}^{T-1} \left[(1-\rho)(1-\nu)\beta_s\right]^{\tau}} + \mu_q \end{aligned}$$